MODULAR TERM REWRITING SYSTEMS AND THE TERMINATION

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1. Introduction

The direct sum of two term rewriting systems (TRSs) \( R_0 \) and \( R_1 \) is confluent if \( R_0 \) and \( R_1 \) are confluent [10], but the direct sum is not necessarily terminating even if \( R_0 \) and \( R_1 \) are terminating [11]; that is, termination is not a "modular" [6] property for general term rewriting systems. Recently, it was proved [9,5,12,4] that for some restricted classes of systems the direct sum is terminating if \( R_0 \) and \( R_1 \) are.

In this paper we present a novel approach to modularity. Rather than considering the union (direct sum) of the sets of rewrite rules, we consider the family (set) of them, introducing a new reduction relation called a modular reduction. One of the significant points is that we can prove a theorem on its termination which holds for general systems.

Let \( \mathcal{V} \) be a set of variables, and \( \mathcal{F} \) be a set of function symbols. The set of terms on \( \mathcal{F} \) and \( \mathcal{V} \) is denoted by \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \).

Let \( \mathcal{F}_1, \ldots, \mathcal{F}_n \) be pairwise disjoint sets of function symbols and let \( \mathcal{F} \) be their union. Then a modular term rewriting system MR on \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) is a family of term rewriting systems \( R_i \) on \( \mathcal{T}(\mathcal{F}_i, \mathcal{V}) \), i.e., \( \text{MR} = \{R_1, \ldots, R_n\} \). Each member \( R_i \) is called a module of MR. Computation in modular term rewriting systems is defined by the following reduction relation:

\[
\begin{align*}
  s & \Rightarrow^* t \iff s \Rightarrow^*_{R_i} t \text{ for some } R_i, 1 < i < n, \\
  s & \Rightarrow_{R_i} t \iff s \Rightarrow^+_t t \text{ and } t \in \text{NF}(R_i),
\end{align*}
\]

where \( \Rightarrow^+_t \) is the transitive closure of \( \Rightarrow_{R_i} \), the reduction relation defined by \( R_i \); \( \text{NF}(R_i) \) is the set of \( R_i \)-normal forms, i.e., normal forms with respect to \( R_i \). In other words, \( s \Rightarrow_{R_i} t \) iff \( t \) can be obtained by repeatedly rewriting \( s \) by the rewrite rules in \( R_i \) until the term is in \( R_i \)-normal form \( t \).

The relation \( \Rightarrow \) is called the modular reduction relation. Note that \( s \Rightarrow_{R_i} t \Rightarrow_{R_j} u \) implies \( i \neq j \).

Theorem 1. There is no infinite sequence \( t_0 \Rightarrow t_1 \Rightarrow \cdots \) of modular reduction.

Note that the result is independent of the termination of each module.

2. Proof of Theorem 1

Definitions 2. The root of a term \( t \), notation \( \text{root}(t) \), is \( f \) if \( t \) is of the form \( f(t_1, \ldots, t_n) \); otherwise, it is \( t \) itself.

Let \( \square \) be an extra constant called a hole. A term \( C \) on \( \mathcal{F} \cup \{\square\} \) and \( \mathcal{V} \) is called a context on \( \mathcal{F} \).

An alien \(^1\) in a term \( t \) is a nonvariable proper subterm of \( t \) which is maximal with respect to the "subterm" relation, such that \( \text{root}(t) \) and \( \text{root}(u) \) belong to distinct sets of function symbols. We write \( t = C[t_1, \ldots, t_n] \) if \( t_1, \ldots, t_n \) are all the aliens in \( t \) (from left to right) and \( C \) is the context obtained by replacing each alien by a hole.

\(^1\) Also called a principal subterm [12].
The alien tree $AT(t)$ of a term $t$ is the tree each node of which is a context on $F_i$ for some $i$ such that:

1. if $t$ has no alien, then $AT(t)$ consists of a single node $t$, the root of the tree;
2. if $t = C[t_1, \ldots, t_n]$, $n > 0$, then $AT(t)$ consists of the root $C$ and the subtrees $AT(t_i)$, $1 \leq i \leq n$.

The rank of a term $t$, notation $\text{rank}(t)$, is the height of the alien tree of $t$.

Example 3. Let $F_1 = \{f, a\}$, $F_2 = \{g, b\}$, and $F = \{h, c\}$. The term $t \equiv f(f(b, a), h(a, c))$ has the two aliens $b$ and $h(a, c)$; thus $t \equiv C[b, h(a, c)]$, where $C \equiv f(f(\emptyset, a), \emptyset)$. The alien tree of $t$ is depicted below, where $\text{rank}(t) = 2$:

```
  f(f(\emptyset, a), \emptyset)
     /   \
    b     h(\emptyset, c)
        /  \
      a
```

Proof of Theorem 1. By induction on $n$, the number of the modules. The case $n = 1$ is trivial.

Induction step: Assume that there is an infinite sequence $t_0 \Rightarrow_{R_{m_1}} t_1 \Rightarrow_{R_{m_2}} \cdots$, where $1 \leq m_i \leq n$, $i \geq 0$. Every infinite subsequence $t_i \Rightarrow_{R_{m_i}} t_{i+1} \Rightarrow_{R_{m_{i+1}}} \cdots$ uses every modular reduction $\Rightarrow_{R_{m_j}}$, $1 \leq j \leq n$, at least once. (Otherwise, it contradicts the inductive hypothesis.) Therefore, every module is used infinitely many times, and we can divide the sequence into infinitely many pieces of finite subsequences $t_p \Rightarrow \cdots \Rightarrow t_{p,i}$, $i \geq 0$, $p_0 = 0$, such that every piece uses all the modules.

Let $r$ be the rank of the initial term $t_0$. The rank of the $i$th in any such sequence is nonincreasing [10]. Consider all the subterms of $t_0$ that are at the bottom-most ($r$th) layer of the alien tree of $t_0$. After the first piece $t_0 \Rightarrow \cdots \Rightarrow t_p$, the $r$th layer (of $t_p$) contains only normal forms, and from that point on will always be in normal form (not hard to see). In general, if all layers below the $i$th are in normal form, then the rewriting brings the $i$th layer to normal form, while preserving that property for lower layers. Thus, after at most $r + 1$ pieces, the rewriting must terminate. □

We say that a modular TRS is terminating (confluent) if the modular reduction relation is Noetherian (confluent).

Corollary 4. A modular TRS is terminating iff every module is terminating.

Corollary 5. A modular TRS is complete iff every module is complete.

Here, completeness means both termination and confluence. Corollary 5 is direct from our theorem on termination and Toyama’s theorem [10] on the confluence of direct sums. Note that the counterexample by Klop and Barendregt (see [11]) shows that the completeness of all the modules does not necessarily imply the completeness of their union (direct sum).

3. Remarks on modularity

In this section we make some remarks on modularity, by comparing from several viewpoints the modular TRSs with the union of TRSs.

Specification view: A TRS $R_i$ specifies the reduction relation $\Rightarrow_{R_i}$. On the other hand, modular TRSs regard a module $R_i$ as a (maybe non-deterministic) function procedure which accepts an $R_i$-reducible term and returns its $R_i$-normal form. A modular TRS specifies such a procedure, which we call the $R_i$-normalizer.

Implementation view: A module (a normalizer) may be implemented in conventional machine codes or in special hardware devices or in anything else, as well as in rewrite rules (plus an interpreter). In this view, the union of sets of rewrite rules is unnatural for combining modules. On the other hand, a modular TRS may be naturally implemented as a collection of the implementations of the modules plus some synchronization mechanism.

Concurrent programming view: Suppose that each module $R_i$, is associated with the “process” executing the following code:

```
  loop-i: “Normalize the term $t$ with respect to $R_i$”
          go to loop-i
```

2
where \( t \) is a shared variable (shared among the processes). Other processes \( R_j, j \neq i \), are not allowed to access \( t \) while \( R_i \) is executing the first line. Thus the first line is a “critical section”, which is a section of code that inhibits interleavings of actions of other processes. If interleavings were allowed, then unintended results (non-termination in this case) might occur. By restricting interleavings in a natural and simple way, modular TRSs provide a sound framework for synthesis.

**Termination proof view:** Each module has its own proof for termination (possibly under some reduction strategy) by any of the proposed proof methods in, say, [1]. Then, how can we effectively combine these proofs to prove the termination of the union? Modular TRSs combine these proofs effectively and implicitly, requiring no more proof.

**Localization view:** Modular TRSs provide three kinds of localization. First, the disjointness of the sets of function symbols obviously means the localization of symbols. Since symbols represent functions logically (or declaratively) defined in the modules, this is the localization of logic. Second, the rewrite rules used in a modular reduction step are localized in a single module; thus, the localization of control. Third, each module may have a local reduction strategy of its own, given a responsibility for the termination of rewritings; so, the localization of strategy.

### 3.1. Related work

Dershowitz [1], Rusinowitch [9], Middeldorp [5], Toyama et al. [12], and Kurihara and Ohuchi [4] presented several results on the termination of a union of some restricted systems such as noncollapsing, nonduplicant, or left-linear systems, etc. (In this paper, however, we considered a family of general systems.) Toyama [10] proved the modularity of confluence of direct sums. Nelson and Oppen [7] presented a method of combining decision procedures of disjoint theories.

### 4. Application—innermost modular reduction

Assume that the computation in every module terminates possibly under appropriate strategies. Then, apparently, the following variant of modular reduction terminates, and computes a normal form of \( t \):

\[
\text{rewrite}(t) \\
(1) \text{ for each alien } t_i \text{ in } t, \quad \text{rewrite}(t_i) \\
(2) \text{ assume that root}(t) \in \mathcal{R}_j. \text{ Then, replace } t \text{ by } u \text{ such that } t \Rightarrow_{\mathcal{R}_j} u
\]

where we assume that a term contained in \( t \) is destructively replaced by the normal form. Note that in (2) the redexes can occur only in the topmost (0th) layer of the current term. Also note that this variant is not equivalent to the ordinary innermost reduction, especially when the local strategies (in (2)) of some modules are not innermost.

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### References


