知能ソフトウェア特論
Intelligent Software

項書換え系（３）
合流性

Term Rewriting Systems(3)
Confluence

1. 抽象書換え系と合流性(1/3)
(Abstract reduction systems and confluence)

■定義

抽象書換え系 (A,R)

任意の集合 A と二項関係 $R \subseteq A \times A$ の対.

R を $\to$ と書く.

$(a,b) \in \to$ のとき, $a \to b$ と書く.

a は b に簡約可能

■Definition

Abstract reduction system (A,R) consists of a set A and a binary relation $R \subseteq A \times A$.

We will write $\to$ for R.

When $(a,b) \in \to$, we write $a \to b$ and say that a is reducible to b.

【Example】

$$A = \{a,b,c,d,e\}$$

$$R = \{(a,b),(b,a),(a,c),(b,d),(e,d)\}$$

$$\to = \{(a,b),(b,a),(a,c),(b,d),(e,d)\}$$

$$\begin{array}{ccccc} & a & \rightleftarrows & b & & e \\ \swarrow & & & & \searrow \; \swarrow & \\ c & & & & d & \end{array}$$

図1

1．抽象書換え系と合流性(2/3)
(Abstract reduction systems and confluence)

■定義

反射推移閉包　$a \to^* b$

a から b へ 0 ステップ以上の $\to$ で書換え可能.

推移閉包　$a \to^+ b$

a から b へ 1 ステップ以上の $\to$ で書換え可能.

会同性　$a \downarrow b$

$a \to^* c$ かつ $b \to^* c$ を満たす $c \in A$ が存在する

■Definition

Reflexive transitive closure $\to^*$:

We write $a \to^* b$, if b can be obtained from a by zero or more steps of reduction by $\to$.

Transitive closure $\to^+$:

We write $a \to^+ b$, if b can be obtained from a by one or more steps of reduction by $\to$.

Joinability $\downarrow$:

a and b are joinable ($a \downarrow b$), if there exists c in A such that $a \to^* c$ and $b \to^* c$.

【Example】

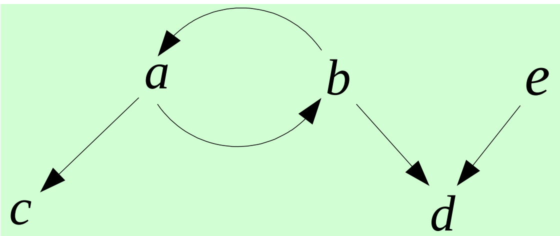


図1

$a \to^* d$, $a \to^* a$

$a \to^+ d$

$a \downarrow e$, $a \downarrow b$

1．抽象書換え系と合流性(3/3)
(Abstract reduction systems and confluence)

■定義

弱合流性

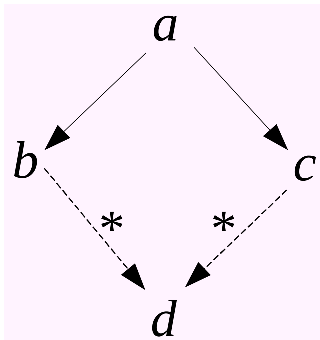
任意の $a,b,c \in A$ に対して，

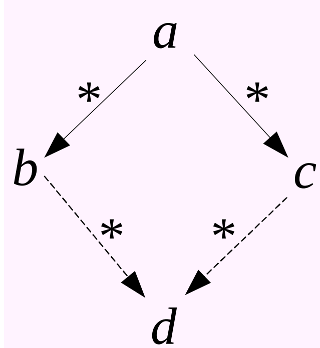
$a \to b$ かつ $a \to c$ ならば $b \downarrow c$

合流性

任意の $a,b,c \in A$ に対して，

$a \to^* b$ かつ $a \to^* c$ ならば $b \downarrow c$





■Definition

Weak confluence:
$(A, \to)$ is weakly confluent,
if for all a, b and c in A,
$a \to b$ and $a \to c$ implies $b \downarrow c$.

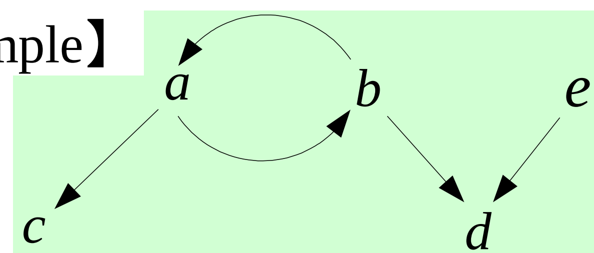
Confluence:
$(A, \to)$ is confluent,
if for all a, b and c in A,
$a \to^* b$ and $a \to^* c$ implies $b \downarrow c$.

合流性をもつならば，弱合流性をもつ．

その逆は成り立たない．

Confluence implies weak confluence, but its converse does not hold as the example in the right figure shows.

【Example】



弱合流性をもつが合流性をもたない例
（c と d が join しない）

２．合流性の基本性質 (1/3)
(Basic properties of confluence)

■定義

$a \in A$ は**正規形**である：

$a \to b$ なる $b \in A$ が存在しない.

$a \in A$ は**正規形をもつ**：

$a \to^{*} b$ なる正規形 $b \in A$ が存在する.

$(A, \to)$ は**一意の正規形をもつ**：

任意の $a \in A$ に対し,

a が高々１つの正規形をもつ.

■**Definition**

$a \in A$ is a normal form,
if there exists no $b \in A$ such that $a \to b$.

$a \in A$ has a normal form,
if there exists a normal form $b \in A$ such that $a \to^{*} b$.

$(A, \to)$ has a unique normal form,
if every $a \in A$ has at most one normal form.

２．合流性の基本性質 (2/3)
(Basic properties of confluence)

■定理（合流性$\Rightarrow$正規形が一意）

$(A,\to)$ は，合流性をもつならば，一意の正規形をもつ

■**Theorem (Confluence ⇒ Unique normal form)**

Every confluent system has a unique normal form.

（証明）

a の２つの正規形を b,c とすると，

$a \to^* b$，$a \to^* c$ および合流性より $b \downarrow c$.

b,c は正規形なので，$b = c$.

(Proof)

If *a* has two normal forms *b* and *c*, then from $a\to^*b$ and $a\to^*c$, we have $b\downarrow c$. Since *b* and *c* are normal forms, it must be the case that $b=c$.

合流性は**関数型プログラム**に望まれる性質
（関数の返す値は非決定的な
並列計算をしても一意）

Confluence is a desirable property for **functional programs**, as the values returned by functions should be unique even under nondeterministic, parallel computation.

２．合流性の基本性質 (3/3)
(Basic properties of confluence)

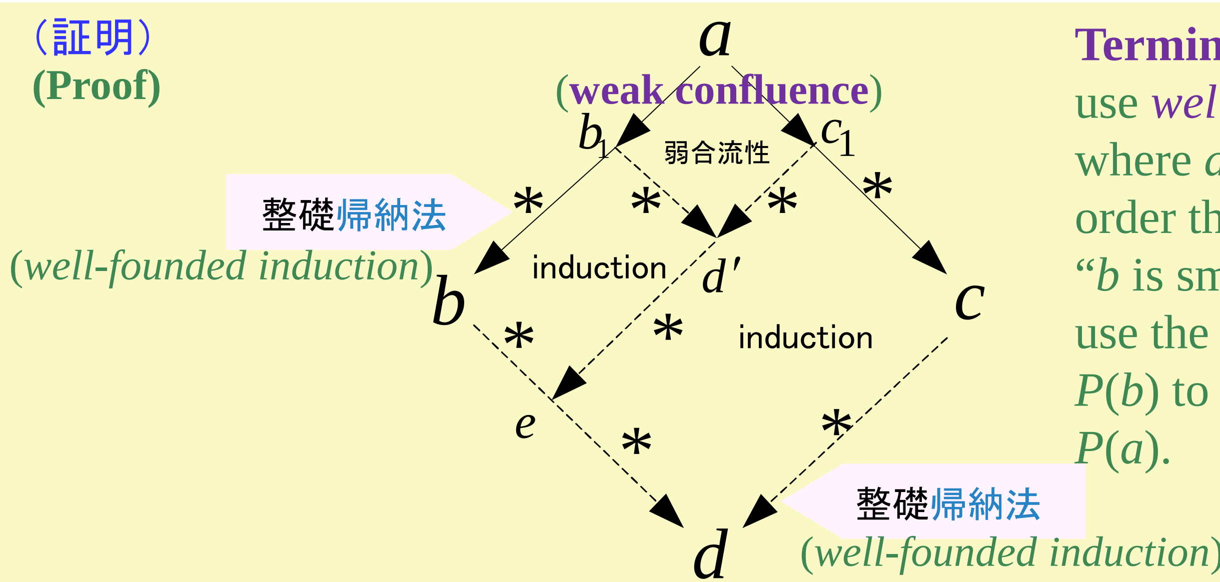
■定理（停止性＋弱合流性 ⇒ 合流性）［ニューマンの補題］

停止性と弱合流性をもつ $(A,\to)$ は合流性をもつ.

■Theorem (Termination + Weak confluence ⇒ Confluence) [Newmann's lemma]

Every terminating, weakly confluent system is confluent.

（証明）
(Proof)



Termination of → allows us to use *well-founded induction*, where $a \to^+ b$ is a strict partial order that can be interpreted as "b is smaller than a", and we can use the induction hypotheses $P(b)$ to prove the induction step $P(a)$.

停止性の検証方法は前回学んだので，合流性を検証するにはあと弱合流性の検証法を学べばよい

Having already studied verification of termination, we will study verification of weak confluence to verify confluence.

【準備】代入(1/4)
(Preliminaries: Substitution)

■定義

代入 $\sigma = \{x_1/s_1, x_2/s_2, \cdots, x_n/s_n\}$

変数 $x_i (i = 1,2,\cdots,n)$ を項 s_i に同時に置き換える操作を表す

$$\sigma(x_i) = s_i$$

■Definition

A substitution $\sigma=\{x_1/s_1,\ldots, x_n/s_n\}$ represents the operation that replaces all the occurrences of variables x_i $(i=1,2,\ldots,n)$ by the terms s_i. We will write $\sigma(x_i)=s_i$.

【Example】 $\sigma = \{x/a, y/g(z)\}$

x	***y***
a	***g(z)***

$$\sigma(x) = a$$

$$\sigma(y) = g(z)$$

As this example shows, a substitution can be well represented as a table that maps each variable to a term.

【準備】代入(2/4)
(Preliminaries: Substitution)

■定義

代入の（項 t への）適用 $\sigma(t)$

項 t 内のすべての変数に対して，σで指定された置き換えを同時に一回行った結果を表す

■Definition

Application of substitution σ to term t is denoted by $\sigma(t)$.

This represents the term obtained from t by replacing all the variables x_i (i=1,2,…,n) by s_i.

【Example】 $\sigma = \{x/a, y/g(z)\}$

$$t = g(x, f(c, x, y), z)$$

x	***y***
a	***g(z)***

$$\sigma(t) = g(a, f(c, a, g(z)), z)$$

【準備】代入(3/4)
(Preliminaries: Substitution)

代入σは，項の集合TからTへの関数
$$\sigma: T \rightarrow T$$
とみなすことができる

A substitution σ can be seen as a function from the set of terms T to T.

■定義　代入の合成 $\sigma_1 \circ \sigma_2$

$$(\sigma_1 \circ \sigma_2)(t) = \sigma_1(\sigma_2(t))$$

■Definition
The composition of two substitutions σ_1 and σ_2 is the substitution $\sigma_1 \circ \sigma_2$ defined by
$$(\sigma_1 \circ \sigma_2)(t) = \sigma_1(\sigma_2(t))$$

【Example】

$$\begin{tikzcd}
& g(x,y) \arrow[dl, "\sigma=\{x/y\}"'] \arrow[dd, "\sigma'=\tau\circ\sigma=\{x/b,\, y/b\}"] \\
g(y,y) \arrow[dr, "\tau=\{y/b\}"'] & \\
& g(b,b)
\end{tikzcd}$$

【準備】代入(4/4)
(Preliminaries: Substitution)

■定義　代入の一般性

代入 σ は代入 σ' より一般的である

$$\Updownarrow$$

ある代入 τ に対し,　$\sigma' = \tau \circ \sigma$

■**Definition**

A substitution σ is more general than a subsitution σ', if there exists a substitution τ such that $\sigma' = \tau \circ \sigma$.

【Example】

$$\begin{tikzcd}
g(x,y) \arrow[d, "\sigma = \{x/y\}"'] \arrow[dd, "\sigma' = \tau \circ \sigma = \{x/b, y/b\}"] \\
g(y,y) \arrow[d, "\tau = \{y/b\}"'] \\
g(b,b)
\end{tikzcd}$$

In this example, σ is more general than σ'.
　Note that, intuitively, the result $g(y,y)$ of σ is more general than the result $g(b,b)$ of σ', as the universally quantified variable y is more general than the specific constant b.

３． 単一化 (1/6) (Unification)

■定義

単一化

s,t に対し，ある代入 σ が存在して

$$\sigma(s) = \sigma(t)$$

とできるとき，

σ を s,t の**単一化代入**といい，

s,t は**単一化可能**であるという．

■Definition

Two terms s and t are unifiable, if there exists a substitution σ, called a unifier of s and t, such that

$$\sigma(s) = \sigma(t).$$

■定義

最汎単一化代入(mgu) σ

単一化代入のうち，**最も一般的**なもの．

（σ よりも一般的な単一化代入がない）

■Definition

The most general unifier (mgu) σ of two terms s and t is a unifier σ such that there exists no unifier more general than σ.

３． 単一化 (2/6) (Unification)

【Example】 $f(x, g(x))$ $f(h(y), z)$

UNIFY

This is not the *mgu*.

$\sigma' = \{x/h(a),\ y/a,\ z/g(h(a))\}$

This is the *mgu*.

$\sigma = \{x/h(y),\ z/g(h(y))\}$

$f(h(a), g(h(a)))$

$\tau = \{y/a\}$

$f(h(y), g(h(y)))$

３．単一化 (3/6) (Unification)

単一化アルゴリズム

【入力】 項 s,t

【出力】 項 s,t が単一化可能ならば *mgu* を出力．
単一化可能でなければ「失敗」を出力．

【手順】 関数記号を解釈しない方程式 $s = t$ を変形し，$x_i = u_i$ の形の複数の方程式に変換して，代入(*mgu*) $\sigma =\{ x_i / u_i \mid 1 \leqq i \leqq n\}$ を構成する．

A unification algorithm accepts two terms *s* and *t* as input, and outputs the their *mgu* if they are unifiable; or the *failure* otherwise.
Its procedure is basically the transformation of the equation $s=t$ (in which the function symbols are not interpreted) into a set of equations of the form $x_i=u_i$, from which the resultant substitution (*mgu*) $\sigma =\{x_i / u_i \mid 1 \leqq i \leqq n\}$ will be constructed.

【Example】 Equation: $\overset{s}{f(x, g(x))} = \overset{t}{f(h(y), z)}$

Solution: $\sigma = \begin{cases} x = h(y) \\ z = g(h(y)) \end{cases}$

３．単一化 (4/6) (Unification)

Step1. 連立方程式$\{s=t\}$に対し，つぎの５つの変換操作を任意に繰り返し適用して変形する．（定数は引数０個の関数記号とみなす.）

Apply arbitrarily the following five transformation operations to the initial set of equations $\{s=t\}$. (Constants are regarded as function symbols with no arguments.)

x is a variable and t is a term.

(1) $f(s_1,\ldots,s_n)=f(t_1,\ldots,t_n)$ $\Rightarrow$ transform to $s_1=t_1,\ldots,s_n=t_n$

(2) $f(s_1,\ldots,s_n)=g(t_1,\ldots,t_m)$ $(f\neq g)$ $\Rightarrow$ return *failure*

(3) $t=x$ $\Rightarrow$ transform to $x=t$

(4) $x=x$ $\Rightarrow$ remove this equation

(5) $x=t$ $(t\neq x)$ $\Rightarrow$ if x occurs in t, then return *failure*

else if x occurs in other equations, then

apply the substitution $\{x/t\}$ to them

Occurrence check:
x = f(x)

３． 単一化 (5/6) (Unification)

Step2.　**Step1** のどの操作も適用できなくなったとき，連立方程式は

$$\{x_1 = u_1, \ldots, x_n = u_n\}$$

の形になっており，左辺の各変数はどの右辺の項の中にも出現していない．このとき s, t は単一化可能で,

$$\sigma = \{x_1/u_1, \ldots, x_n/u_n\}$$

が *mgu* である．

When no operations of Step 1 are applicable any more, the set of equations should be in the form

$$\{ x_i = u_i \mid 1 \leqq i \leqq n \}$$

and no variables in the left-hand sides occur in the right-hand sides.

In this case, *s* and *t* are unifiable and their *mgu* is

$$\sigma = \{ x_i / u_i \mid 1 \leqq i \leqq n \}$$

３． 単一化 (6/6) (Unification)

【Example】Unify $f(x, g(x))$ and $f(h(y), z)$.

$$\{ f(x, g(x)) = f(h(y), z) \}$$

$$\Rightarrow \begin{cases} x = h(y) \\ g(x) = z \end{cases}$$

$$\Rightarrow \begin{cases} x = h(y) \\ z = g(x) \end{cases}$$

$$\Rightarrow \begin{cases} x = h(y) \\ z = g(h(y)) \end{cases}$$

$$\sigma = \{x/h(y), z/g(h(y))\}$$

４． 危険対による合流性の判定 (1/5)

(Decision on confluence by critical pairs)

【動機】２つのルール

$$f(f(x,y),z) \to f(x, f(y,z))$$

$$f(i(w),w) \to e$$

のどちらでも書換え可能な一般性のある重なりを求めるため，１つめのルールの左辺の部分項と２つめの左辺の全体の単一化を試す．

【Motivation】 Consider two rules

$$f(f(x, y), z) \to f(x, f(y, z))$$

$$f(i(w), w) \to e.$$

To find general overlaps which can be reduced by any of them, try to unify a subterm of the left-hand side (LHS) of the first rule and the whole LHS of the second rule.

$f(x,y)$ and $f(i(w),w)$ are unifiable with *mgu* $\sigma = \{x/i(w), y/w\}$.

重なり(overlap)

σ(left-hand side of first rule)$=$ $f(\underline{f(i(w),w)},z)$

弱合流性を乱す可能性のパターン
(This pattern might violate weak confluence.)

危険対
(critical pair)

$f(e,z)$ $\qquad$ $f(i(w), f(w,z))$

4. 危険対による合流性の判定 (2/5)
(Decision on confluence by critical pairs)

$l_1 \to r_1$, $l_2 \to r_2$：互いに共通の変数を持たないように適切に変数名を付け替えてある２つの書換え規則

■定義　危険対

$l_1[s] \to r_1$　（s は l_1 の部分項で非変数）

$l_2 \to r_2$

s と l_2 は単一化可能： $\sigma(s) = \sigma(l_2)$

危険対

$$\langle \sigma(l_1)[\sigma(r_2)], \sigma(r_1) \rangle$$

overlap

$$\sigma(l_1[s]) = \sigma(l_1)[\sigma(s)]$$

critical pair

$$\boxed{\sigma(l_1)[\sigma(r_2)] \qquad \sigma(r_1)}$$

Let $l_1 \to r_1$ and $l_2 \to r_2$ be two rules in which variables are renamed so that they share no variables in common.

■Definition

A critical pair is a pair of terms **σ(l_1)[σ(r_2)]** and **σ(r_1)**, where σ is the *mgu* of l_2 and a non-variable subterm s of l_1 (i.e., $\sigma(s) = \sigma(l_2)$).

The notation $l_1[s]$ emphasizes that l_1 contains s as its subterm.

σ(l_1)[σ(r_2)] represents the term obtained from the overlap **σ(l_1)[σ(s)]** by replacing the subterm $\sigma(s)$ (=$\sigma(l_2)$) by σ(r_2).

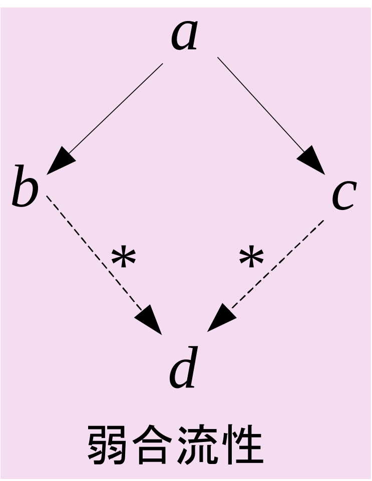
4. 危険対による合流性の判定 (3/5)
(Decision on confluence by critical pairs)

■定理（危険対定理）

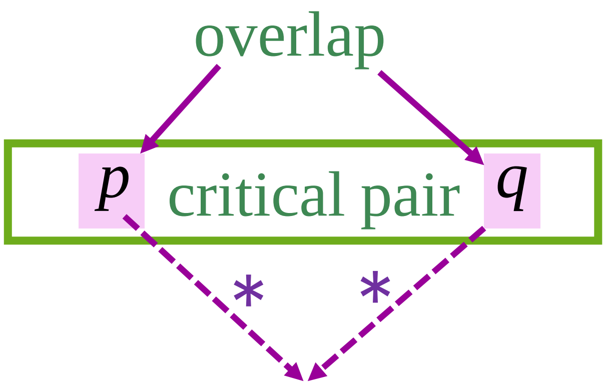
項書換え系 R が弱合流性を満たすための必要十分条件は，R のすべての危険対 $\langle p,q \rangle$ が会同すること $(p \downarrow q)$ である．

■Theorem (Critical pair theorem)

A term rewriting system R is weakly confluent if, and only if, every critical pair $\langle p,q \rangle$ is joinable, i.e. $p{\downarrow}q$.



(weak confluence)



(有限個の) 危険対だけ考えればよい

We need to think about only a *finite* number of critical pairs to verify the weak confluence.

4． 危険対による合流性の判定 (4/5)

(Decision on confluence by critical pairs)

系（停止性 ⇒ （危険対による合流性判定））

停止性をもつ項書換え系 R が合流性を満たすための必要十分条件は, R のすべての危険対 $\langle p,q \rangle$ について p と q の正規形が一致することである.

■Corollary (Confluence check by critical pairs for terminating TRS)

A terminating, term rewriting system R is confluent if, and only if, for all critical pairs $\langle p,q \rangle$, normal forms of p and q are identical.

（証明） ニューマンの補題（停止性∧弱合流性⇒合流性）と危険対定理を組み合わせる.

(Proof) Combine the Newmann's lemma with the Critical pair theorem.

Algorithm

Step 1. Let S = the finite set of all critical pairs of R.

Step 2. For each critical pair $\langle p,q \rangle$ **in** S

  Let p^* = a normal form of p; and q^* = a normal form of q.

  If $p^* \neq q^*$, **then return** *false* (R is *not* confluent).

Step 3. Return *true* (R is confluent).

４． 危険対による合流性の判定 (5/5)

(Decision on confluence by critical pairs)

【Example】

$$R = \begin{cases} f(g(x)) \to h(x,x) \\ g(e) \to e \\ f(e) \to h(e,e) \end{cases}$$

停止性：あり．

It is not hard to show that R is terminating.

危険対： $f(e) \leftarrow \underline{f(\underline{g(e)})} \to h(e,e)$ から得られる

$\langle f(e), h(e,e) \rangle$．

R has only one critical pair $\langle f(e),h(e,e)\rangle$.

２つの項の正規形はともに $h(e,e)$ なので，

R は合流性を満たす．

Since the normal forms of both $f(e)$ and $h(e,e)$ are $h(e,e)$, i.e., the same, R is confluent.

演習問題 7
EXERCISE 7

停止性を満たすつぎの項書換え系 R が合流性をもつことを示せ．（危険対は２つある．）

Verify that the following terminating TRS R is confluent. (Hint: R has two critical pairs.)

$$R = \begin{cases} plus(x, 0) & \to & x \\ plus(0, y_1) & \to & y_1 \\ plus(s(x_2), y_2) & \to & s(plus(x_2, y_2)) \end{cases}$$

また，つぎの項書換え系 S は合流性をもたないことを示せ．（１番目と２番目の書換え規則から得られる２つの危険対のうちの一方において，正規形が一致しない．）

Verify that the following TRS S is *not* confluent. (Hint: The first two rules have two critical pairs, one of which consists of two terms that have normal forms different from each other.)

$$S = \begin{cases} plus(x, plus(y, z)) & \to & plus(plus(x, y), z) \\ plus(0, y_1) & \to & y_1 \\ plus(s(x_2), y_2) & \to & s(plus(x_2, y_2)) \end{cases}$$